

Tripoli university
Faculty of engineering
EE department EE313
Electrical polarization tutorial

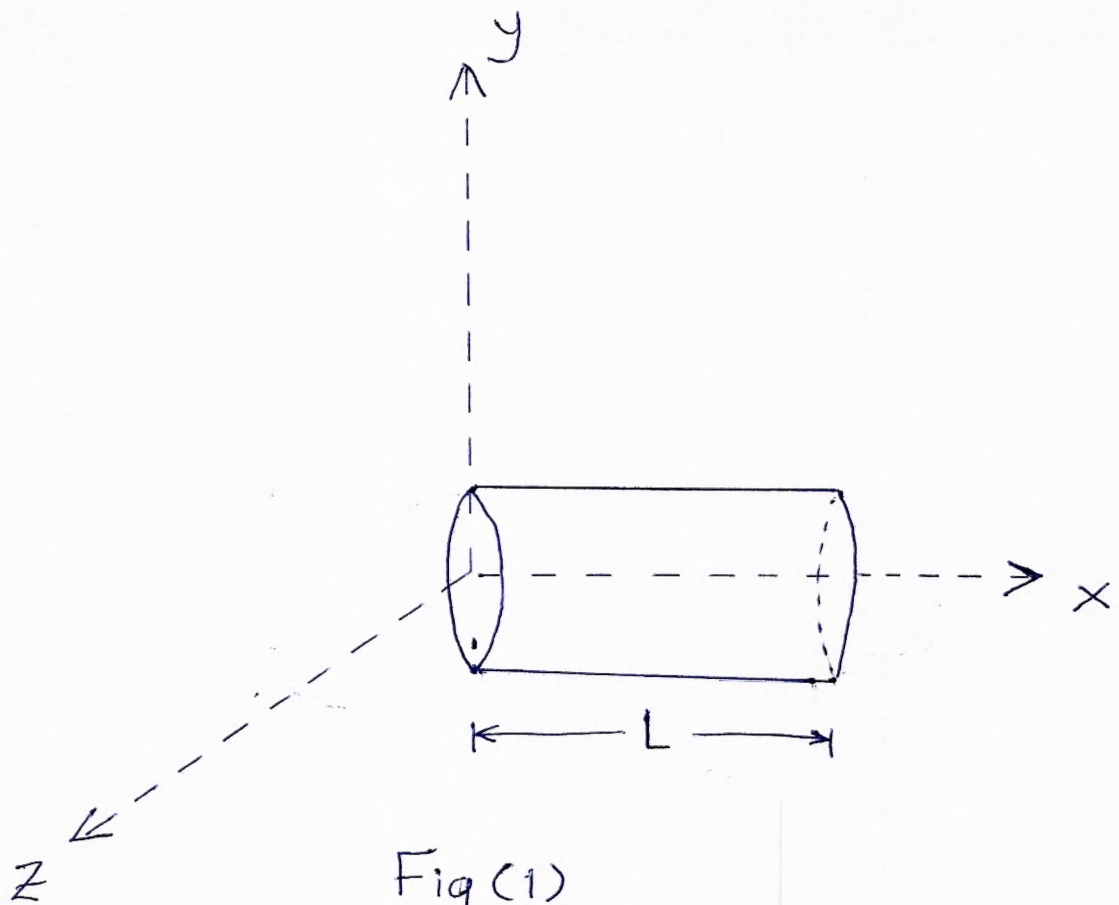
Problem #1

A thin rod of cross section A extends along the x -axis from $x=0$ to $x=L$. The polarization of the rod is along its length and is given by $P_x = ax^2 + b$. Calculate ρ_p and ρ_{sp} at each end. Show that the total bound charge vanishes in this case.

Solution

$$\rho_p = -\vec{\nabla} \cdot \vec{P} = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) = -2ax$$

We can sketch the rod in fig(1)



At first end of the rod ($x=0$):

$$\rho_p = -2a(0) = 0$$

At the other end of the rod ($x=L$):

$$\rho_p = -2aL$$

From boundary conditions of the polarization vector when region 1 is free space (see equation 3-47):

$$\vec{n} \cdot \vec{P}_2 = \rho_{sp}$$

where \vec{n} is a unit vector normal to the surface separating the two regions and pointing into region 1.

\vec{n} in our problem is $-\vec{a}_x$ at $x=0$ and \vec{n} is \vec{a}_x at $x=L$.

at surface $x=0$:-

$$\rho_{sp} = -(ax^2 + b)_{x=0} = -b$$

at surface $x=L$:-

$$\rho_{sp} = (ax^2 + b)_{x=L} = aL^2 + b$$

Now, we'll prove that the total charge on the rod must be zero:

$$Q_t = \int_V \rho_p dv + \int_{s \text{ at } x=0} \rho_{sp} ds + \int_{s \text{ at } x=L} \rho_{sp} ds$$

$$= \int_0^L \rho_p A dx + \rho_{sp} A_{\text{at } x=0} + \rho_{sp} A_{\text{at } x=L}$$

$$= -A(2a) \int_0^L x dx - bA + aL^2 A + bA$$

$$= -2Aa \left[\frac{x^2}{2} \right]_0^L + aL^2 A$$

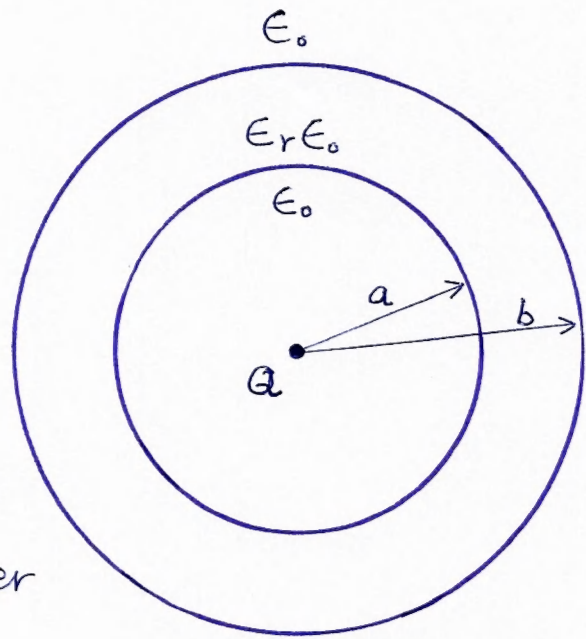
$$= aL^2 A - aL^2 A = 0$$

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Problem #2

Consider the fig. as a spherical dielectric shell so that $\epsilon = \epsilon_0 \epsilon_r$ for $a < r < b$ and $\epsilon = \epsilon_0$ for $0 < r < a$. If a charge Q is placed at the center of the shell, find:-



- a) \vec{P} for $a < r < b$. b) ρ_p for $a < r < b$. c) ρ_{sp} at $r=a, r=b$.

Solution

Applying Gauss law for materials (equation 3-37):

$$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho_v dv$$

where the surface to be chosen is a sphere with Q at its center and radius r .

where $\int_V \rho_v dv$ is simply the charge enclosed by the surface which is the point charge Q .

Hence, :-

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r r^2 \sin\theta d\theta d\phi = Q$$

$$4\pi r^2 D_r = Q$$

$$\therefore \vec{D} = \vec{a}_r \frac{Q}{4\pi r^2}$$

which is valid for any region.

Now, \vec{E} in the dielectric is from equation (3-30b):-

$$\vec{E} = \frac{\vec{D}}{\epsilon_r \epsilon_0} = \vec{a}_r \frac{Q}{4\pi \epsilon_r \epsilon_0 r^2}$$

And \vec{P} is found within the dielectric using equation (3-23) :-

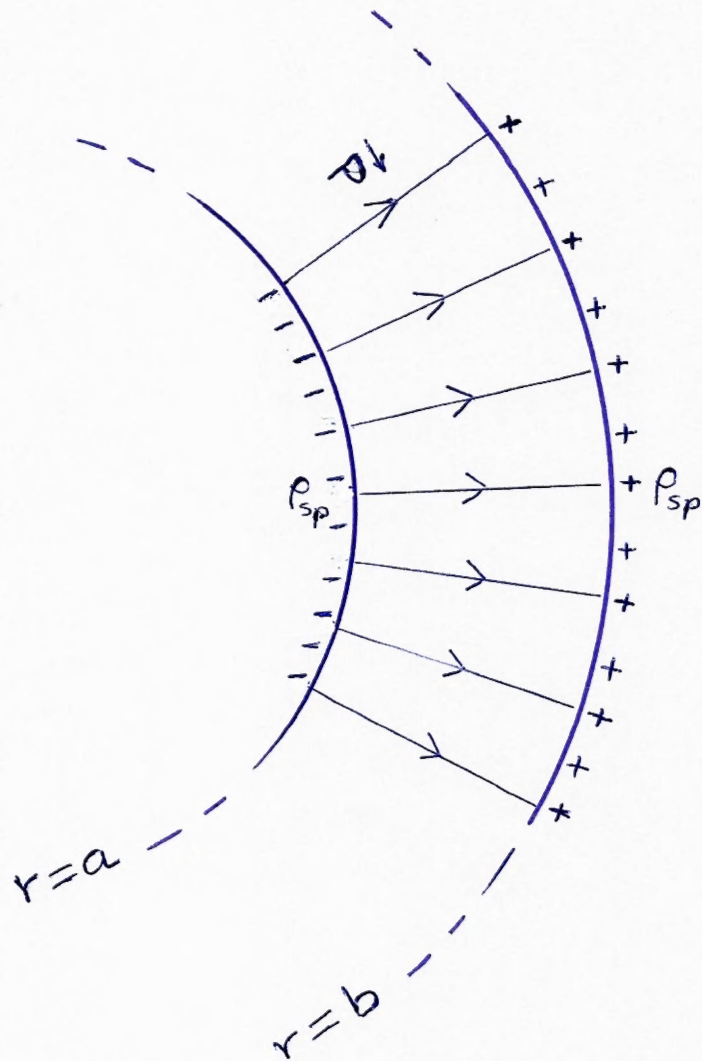
$$\begin{aligned} \vec{P} &= \vec{D} - \epsilon_0 \vec{E} = \vec{a}_r \left[\frac{Q}{4\pi r^2} - \frac{Q}{4\pi \epsilon_r r^2} \right] \\ &= \vec{a}_r \frac{Q}{4\pi r^2} \left(1 - \frac{1}{\epsilon_r} \right) \end{aligned}$$

$$b) \rho_p = -\vec{\nabla} \cdot \vec{P}$$

$$= - \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r P_r) + 0 + 0 \right]$$

$$= -\frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{Q}{4\pi} \left(1 - \frac{1}{\epsilon_r} \right) \right) = 0$$

c)



Fig(2)

For boundary $r=a$:-

for region 1 (free space) and region 2 (dielectric)

$$\vec{n} = -\vec{a}_r$$

$$\rho_{sp} = \vec{n} \cdot \vec{P}_2 = - \left[\frac{Q}{4\pi r^2} \left(1 - \frac{1}{\epsilon_r} \right) \right]_{r=a} = - \frac{Q}{4\pi a^2} \left(1 - \frac{1}{\epsilon_r} \right)$$

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For boundary $r=b$:-

For region (1) (free space) and region (2) (dielectric)

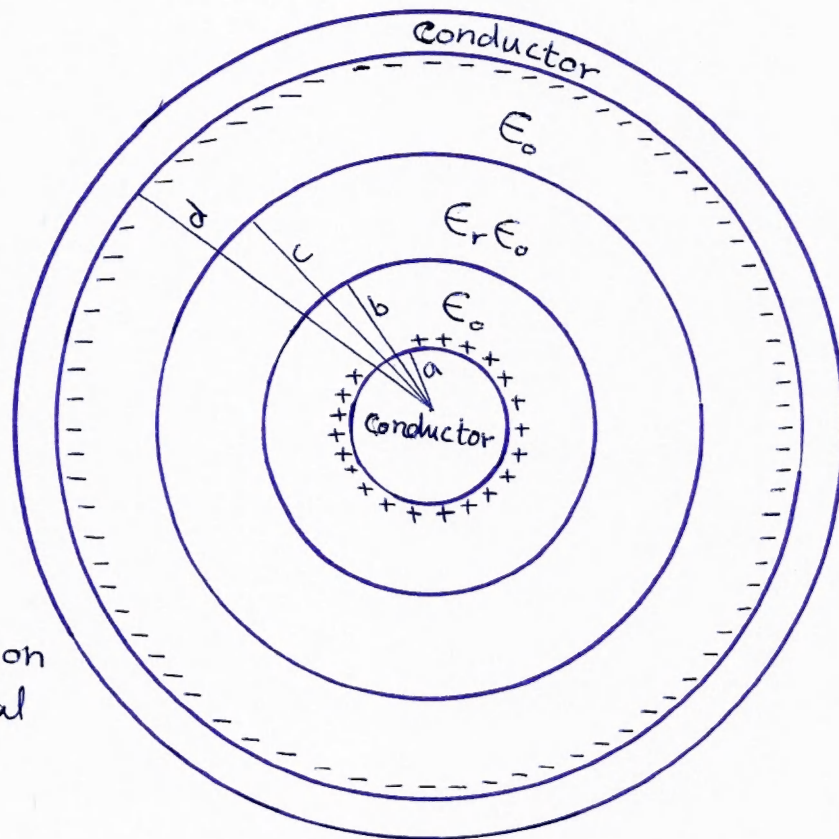
$\vec{n} = \vec{a}_r$. from equation 3-47 :-

$$P_{sp} = \vec{n} \cdot \vec{P}_2 = \left[\frac{Q}{4\pi r^2} \left(1 - \frac{1}{\epsilon_r} \right) \right]_{r=b} = \frac{Q}{4\pi b^2} \left(1 - \frac{1}{\epsilon_r} \right).$$

Problem #3

This problem is (3-11) in Johnk.

Solution



Fig(3)

Cross section
of the coaxial
pair.

If we apply Gauss law (3-37) by choosing the surface as a cylinder with radius ρ and $a < \rho < d$:-

$$\int_{z=0}^l \int_{\phi=0}^{2\pi} D_{\rho} \rho d\phi dz = Q$$

where the total charge on the inner conductor for length l is given in the question to be Q .

$$2\pi \rho l D_{\rho} = Q$$

$$\therefore \vec{D} = \vec{a}_{\rho} \frac{Q}{2\pi \rho l}$$

which is valid in dielectric region as well as in free-space regions.

b)

In the dielectric region ($b < \rho < c$):-

$$\vec{E} = \frac{\vec{D}}{\epsilon_r \epsilon_0} = \vec{a}_{\rho} \frac{Q}{2\pi \epsilon_r \epsilon_0 \rho l}$$

and the polarization \vec{P} :-

$$\begin{aligned}\vec{P} &= \vec{D} - \epsilon_0 \vec{E} \\ &= \vec{a}_\rho \left[\frac{Q}{2\pi\rho\ell} - \frac{Q}{2\pi\epsilon_r\rho\ell} \right] \\ &= \vec{a}_\rho \frac{Q}{2\pi\rho\ell} \left(1 - \frac{1}{\epsilon_r} \right)\end{aligned}$$

~~$$\rho_p = -\vec{\nabla} \cdot \vec{P} = -\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho P_\rho) + 0 + 0 \right]$$~~

$$\begin{aligned}\rho_p &= -\vec{\nabla} \cdot \vec{P} = -\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho P_\rho) + 0 + 0 \right] \\ &= -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{Q}{2\pi\ell} \left(1 - \frac{1}{\epsilon_r} \right) \right) = 0\end{aligned}$$

c)

For the boundary $\rho=a$ (see fig(3)) we have conductor and free space. If we apply the boundary conditions of the D vectors by choosing the conductor to be region(2) and the free space to be region(1) then $\vec{n} = \vec{a}_\rho$. From equation (3-45) :-

$$\rho_s = \vec{n} \cdot \vec{D}_1 = \left[\frac{Q}{2\pi\rho\ell} \right]_{\rho=a} = \frac{Q}{2\pi a\ell}$$

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At boundary $\rho=d$, let the conductor to be region(2) and the free space to be region(1) again where $\vec{n} = -\vec{a}_\rho$:-

$$\rho_s = \vec{n} \cdot \vec{D} = - \left[\frac{Q}{2\pi\rho l} \right]_{\rho=d} = - \frac{Q}{2\pi d l}$$

For the boundary $\rho=b$, let region(1) to be the free space and region(2) to be the dielectric. By applying the boundary conditions of the \vec{P} vectors (equation 3-47) where $\vec{n} = -\vec{a}_\rho$:

$$\rho_{sp} = \vec{n} \cdot \vec{P}_2 = - \left[\frac{Q}{2\pi\rho l} \left(1 - \frac{1}{\epsilon_r}\right) \right]_{\rho=b} = - \frac{Q}{2\pi b l} \left(1 - \frac{1}{\epsilon_r}\right)$$

For the boundary $\rho=c$, let region(1) to be the free space and region(2) to be the dielectric ($\vec{n} = \vec{a}_\rho$):

$$\rho_{sp} = \vec{n} \cdot \vec{P}_2 = \left[\frac{Q}{2\pi\rho l} \left(1 - \frac{1}{\epsilon_r}\right) \right]_{\rho=c} = \frac{Q}{2\pi c l} \left(1 - \frac{1}{\epsilon_r}\right)$$

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